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# Optimised renormalisation group transformations of lattice spin models

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**Abstract.** The optimisation of real space renormalisation group transformations is explored in the framework of series expansions. A block-spin transformation (in addition to decimation) is constructed which exactly maps the one-dimensional nearest-neighbour Ising model onto itself. The same construction can also be applied to the $q$-state Potts model. In higher dimensions, similar optimisation can be achieved only if the weight function has interactions of range comparable to those generated in the series expansion.

## 1. Introduction

A renormalisation group (RG) transformation consists of a coarse graining procedure accompanied by a dilatation. In real space the RG transformation is described by

$$\exp(-H'[\sigma']) = \sum_{\substack{\text{config.} \\ \sigma}} W[\sigma', \sigma] \exp(-H[\sigma]) \tag{1.1}$$

where $H[\sigma]$ is the Hamiltonian for the original spin system and $H'[\sigma']$ is an effective Hamiltonian for a system containing a fewer number of 'block spins'. (The Hamiltonians contain the temperature dependence, $1/kT$, implicitly.) The sum is over all configurations of the original spins and $W[\sigma', \sigma]$ is the weight function for the transformation.

The coarse graining procedure (i.e. the form of $W[\sigma', \sigma]$) is not uniquely defined and may be chosen, to some degree, for convenience. The transformation must, however, preserve the partition function if the long-ranged behaviour is not to be lost. Furthermore the weight function must be sufficiently local so that the transformation is practicable (we will see that this is an important constraint in discussing the two-dimensional Ising model). A non-local weight function may also produce a singular transformation which would distort the long-ranged behaviour. Changing the form of the weight function will cause a shift in the position of the fixed point Hamiltonian within the space of coupling constants. However, provided the weight functions preserves the correct long-range behaviour the fixed point will only be shifted in the direction of redundant operators (Wegner 1976). It is particularly advantageous if the redundant operators are chosen so that the fixed point Hamiltonian is as close as possible to the initial Hamiltonian since this eliminates much of the transient flow needed to reach the critical point. Optimisation to eliminate this transient flow is very important, for instance, in the Monte Carlo renormalisation group (MCRG) where large

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systematic errors occur when the blocked spin Hamiltonian is far from the fixed point (see, for example, Pawley *et al* 1984).

Swendsen (1984b) has suggested it may be possible to optimise a renormalisation group transformation so that a nearest-neighbour Ising model is mapped onto itself. This is the best optimisation we could wish for; however, we will see it does not always preserve the large-scale physics correctly. To find an optimised RG transformation Swendsen proposed using a multi-parameter weight function of the form:

$$W[\sigma', \sigma] = \prod_{n'} \frac{\exp(\sigma'_{n'} \Sigma_\alpha \rho_\alpha R_{\alpha,n'}[\sigma])}{2 \cosh \Sigma_\alpha \rho_\alpha R_{\alpha,n'}[\sigma]}. \tag{1.2}$$

The product is over all the blocked spin lattice sites. The $R_{\alpha,n'}[\sigma]$ are functions, defined at a blocked site $n'$, whose values depend on the original spins in the neighbourhood of that site. Thus, for example, $R_{0,n'}[\sigma]$ is the 'majority rule' operator defined by:

$$R_{0,n'}[\sigma] = \begin{cases} 1 & \text{if } \sum_{i \in n'} \sigma_i > 0 \\ 0 & \text{if } \sum_{i \in n'} \sigma_i = 0 \\ -1 & \text{if } \sum_{i \in n'} \sigma_i < 0 \end{cases} \tag{1.3}$$

while $R_{1,n'}[\sigma]$ depends on the spins in the nearest-neighbour block, etc. The $\rho_\alpha$ are a set of parameters to be chosen for convenience. If $\rho_0$ is made infinite and all the other parameters are set to zero we obtain the usual majority rule transformation. Swendsen noted that the fixed point Hamiltonian for a majority rule transformation has a nearest-neighbour coupling constant which is larger than all the other couplings (Swendsen 1984a). He, therefore, suggested that if $\rho_0$ was made infinite with the other parameters chosen appropriately it may be possible to make the nearest-neighbour Ising model the fixed point of the renormalisation group transformation. (This is equivalent to using a sophisticated tie-breaker.) I have found that allowing $\rho_0$ to vary can be important in finding the most local, optimised, weight functions. Swendsen discussed his procedure in the context of MCRG. Although this is a computationally very efficient technique it does not give much insight into the physical limitations of optimising an RG transformation.

To understand this problem I have examined the optimisation of RG transformations for some simple, one- and two-dimensional, lattice spin models in the framework of high-temperature expansions. In the following section I will derive a block-spin transformation which maps a nearest-neighbour spin-$\frac{1}{2}$ Ising model onto itself. Ironically, the weight function for this transformation is of the form suggested by Swendsen but where $\rho_0$ varies with temperature and all other $\rho_\alpha$ are set to zero. This transformation is then generalised to allow for larger block sizes and finally adapted for the $q$-state Potts model. For all the cases discussed in this section the new coupling constants are related to the old ones in the same way as predicted by the equivalent decimation.

In § 3, we discuss the problems which arise when Swendsen's transformation is applied to the two-dimensional Ising model. In this case the weight function is found to be far from local. This may cause the long distance physics to be corrupted at each application of the RG transformation. Such a transformation would give incorrect predictions for the critical exponents. The details of the calculation for the two-dimensional case are given in the appendix.

## 2. One-dimensional models

We begin this section by considering optimised RG transformations for one-dimensional spin-$\frac{1}{2}$ Ising models. The simplest RG transformation we can apply to this model is decimation. A generalisation of this procedure, introduced by Kadanoff, which is applicable to a larger class of models is given by

$$W[\sigma', \sigma] = \prod_{n'} \tfrac{1}{2}(1 + \tau \sigma'_{n'} \sigma_{2n'}) \tag{2.1}$$

where the $\sigma'_{n'}$ are the new spin variables and the $\sigma_{2n'}$ form the set of original spins which lie on even lattice sites only. $\tau$ is a parameter which must be chosen so that at the critical point: $\tau = 2^{(d-2+\eta)/2}$. This condition arises because equation (2.1) is linear in the sense that the correlation functions for the block spins are linearly related to the correlation functions of the original spins. For the one-dimensional Ising model $d-2+\eta = 0$, thus, $\tau$ must be one at the pseudo-critical temperature ($T = 0$). If $\tau = 1$ at all temperatures then equation (2.1) becomes

$$W[\sigma', \sigma] = \prod_{n'} \delta_{\sigma'_{n'}, \sigma_{2n'}}. \tag{2.2}$$

This is just decimation. It is well known that decimation maps a one-dimensional nearest-neighbour Ising model with coupling $K$ onto another nearest-neighbour Ising model with coupling strength $K'$ satisfying $\omega' = \omega^2$, where $\omega' = \tanh K'$ and $\omega = \tanh K$. However, if $\tau$ is allowed to vary with temperature then new, non-local, couplings would be generated away from the fixed point. Thus decimation can be considered as the exactly optimised form of the more general RG transformation of equation (2.1). For Ising models in higher dimensions $d-2+\eta \neq 0$ so the weight function of equation (2.1) must be used. However, this gives poor numerical results because of the large transient flow required in reaching the neighbourhood of the fixed point (see, for example, Wilson 1975). A better coarse graining procedure which shows less transient flow uses a block spin which depends on all the spins in the block (see, for example, Subbarao 1976).

It is useful to examine one-dimensional RG transformations which treat all the spins on an equal footing. The simplest one-dimensional block-spin operator which does this takes the form $R_{0,n'}[\sigma] = (\sigma_{2n'} + \sigma_{2n'+1})/2$. Notice that this has the same property as the majority rule operator described in equation (1.3). Using this block-spin operator we can construct a one-parameter weight function with an analogous form to Kadanoff's decimation:

$$W[\sigma', \sigma] = \prod_{n'} \tfrac{1}{2}(1 + \tau \sigma'_{n'} R_{0,n'}[\sigma]). \tag{2.3}$$

Notice that this transformation is exactly of the form suggested by Swendsen (equation (1.2)) but with $\tanh \rho_0 = \tau$ and all other $\rho_\alpha$ set to zero. Once again we require that at the pseudo-critical point $\tau = 1$.

To find the effective Hamiltonian $H'[\sigma']$ we first expand $\exp\{-H[\sigma]\}$ as a high-temperature series;

$$\exp(-H[\sigma]) = \cosh {}^{M}K \sum_{i=1}^{M} (1 + \omega \sigma_i \sigma_{i+1}) \tag{2.4}$$

where we have assumed we have $M$ spin with periodic boundary conditions ($\sigma_{M+1} = \sigma_1$)

and $\omega = \tanh K$. Then:

$$\exp(-H'[\sigma']) = \sum_{\substack{\text{config.}\\ \sigma}} \prod_{n'=1}^{M/2} \tfrac{1}{2}(1+\tau\sigma'_{n'}R_{0,n'}[\sigma]) \exp(-H[\sigma])$$

expanding the weight function and using the high-temperature expansion of equation (2.4):

$$\exp(-H'[\sigma']) = 2^{-M/2}\cosh^M K \times \sum_{\substack{\text{config.}\\ \sigma}} \left(1+\tau\sum_{n'}\sigma'_{n'}R_{0,n'}[\sigma]+\tau^2\sum_{m'<n'}\sigma'_{m'}\sigma'_{n'}R_{0,m'}[\sigma]R_{0,n'}[\sigma]+\ldots\right) \times \prod_{i=1}^{M}(1+\omega\sigma_i\sigma_{i+1})$$

using the properties of the spin; $\Sigma_{\sigma_i}\sigma_i = 0$, $\Sigma_{\sigma_i}\sigma_i^2 = 2$, $\Sigma_{\sigma_{2n'}}\sigma_{2n'}R_{0,n'}[\sigma] = 1$, etc, we can evaluate the sum over configurations to give

$$\exp(-H'[\sigma']) = 2^{M/2}\cosh^M K\left(1+\tfrac{1}{4}\tau^2(\omega+2\omega^2+\omega^3)\sum_{n'}\sigma'_{n'}\sigma'_{n'+1} + \tfrac{1}{4}\tau^2(\omega^3+2\omega^4+\omega^5)\sum_{n'}\sigma'_{n'}\sigma'_{n'+2}+\ldots + \tfrac{1}{16}\tau^4(\omega+2\omega^2+\omega^3)^2\sum_{m'+1>n'}\sigma'_{m'}\sigma'_{m'+1}\sigma'_{n'}\sigma'_{n'+1}+\ldots\right). \tag{2.5}$$

Thus we see that, in general, $H'[\sigma']$ will contain non-local interactions. However, if we choose $\tau$ to be

$$\tau = \frac{2\sqrt{\omega}}{1+\omega} \tag{2.6}$$

then the right-hand side of equation (2.5) has the form, up to a multiplicative constant, of a high-temperature expansion for a nearest-neighbour Ising model with a coupling strength $K'$ given by $\omega' = \omega^2$ ($\omega'$ and $\omega$ are given above). Furthermore, this is true at all orders and thus this is an exact result.

Notice that at the pseudo-critical temperature ($T = 0$): $2\sqrt{\omega}/(1+\omega) = 1$ and the transformation is just a majority rule transformation with a random tie-breaker. If $\tau = 1$ at all temperatures then the critical fixed point Hamiltonian is again that of a nearest-neighbour Ising model, but new couplings will be generated as the system flows away from the critical point. This flow away from the fixed point can be constrained to lie in the direction of the nearest-neighbour coupling alone by insisting $\tau$ varies with temperature according to equation (2.6). In this case we have the unusual situation of the RG transformation having an explicit temperature dependence. In the high-temperature limit the weight function becomes equal to a constant ($W[\sigma', \sigma] = 2^{-M/2}$) and hence the new spins do not depend on any of the old spins in this limit.

We can extend this mapping to larger block sizes. For a block of $N$ spins the operator $R_{0,n'}[\sigma]$ takes the form $R_{0,n'}[\sigma] = (\Sigma_{N(n'-1)<i\leq Nn'}\sigma_i)/N$. To make the new Hamiltonian take the form of a nearest-neighbour Ising model $\tau$ must now satisfy the equation:

$$\tau^2 = \frac{N^2\omega^N}{\omega+2\omega^2+\ldots+N\omega^N+(N-1)\omega^{N+1}+\ldots+\omega^{2N-1}} \tag{2.7}$$

We obtain the same result as a decimation where we keep every $N$th spin, namely $\omega' = \omega^N$. At the critical temperature the transformation is not a majority rule but rather the block spin will be in the same direction as the majority of the spins with some probability which depends on the size of the majority. Again in the high-temperature limit the weight function is just equal to a constant (this time: $W[\sigma', \sigma] = 2^{-M/N}$).

We can construct similar transformations for other one-dimensional discrete spin models. For example we can easily extend the above analysis to the $q$-state Potts model with a nearest-neighbour coupling strength $2K$. We can construct a high-temperature expansion in exactly the way as we did for the Ising model except with $\omega = \tanh K$ replaced by $\nu = (e^{2K} - 1)/(e^{2K} + q - 1)$. The analogous weight function to that for the Ising model (equation (2.3)) is

$$W[\sigma', \sigma] = \prod_{n'} q^{-1}(1 + \tau P_{0,n'}[\sigma', \sigma]) \tag{2.8}$$

where $P_{0,n'}[\sigma', \sigma] = (q\delta_{\sigma'_{n'}, \sigma_{2n'}} + q\delta_{\sigma'_{n'}, \sigma_{2n'+1}} - 2)$ is analogous to the term $\sigma'_{n'} R_{0,n'}[\sigma]$ and has the property, $\Sigma_{\text{config.},\sigma'} P_{0,n'}[\sigma', \sigma] = 0$, which is required to preserve the partition function. The new Hamiltonian will have the form of a nearest-neighbour Potts model provided $\tau$ takes the value:

$$\tau = \frac{2\sqrt{\nu}}{1 + \nu}. \tag{2.9}$$

Again the new coupling constant is related to the old one by the decimation result $\nu' = \nu^2$ where $\nu = (e^{2K'} - 1)/(e^{2K'} + q - 1)$ and $\nu$ is given above. The Ising model result is just a special case of this with $q = 2$.

## 3. Higher dimensions

We have found that optimisation is very fruitful in one dimension. It is therefore natural to try to extend these ideas to higher dimensions where they may be of considerable importance. However, in one dimension optimisation is relatively straightforward because the correlation functions scales very simply (e.g. for the Ising model $\langle \sigma(N), \sigma(0) \rangle = \omega^N$). In two dimensions the correlation functions contain complicated combinatoric coefficients which arise because there are now many different paths between two spins on the lattice. Therefore to exactly optimise the transformation we must also include the non-local operators ($R_{\alpha,n'}[\alpha]$; $\alpha > 0$) in Swendsen's weight function.

Using a high-temperature expansion we can construct a weight function, order by order, which removes all but the nearest-neighbour interaction. Thus, for example, using a $2 \times 2$ block spin on a square lattice, majority rule will generate a next-to-nearest-neighbour interaction between diagonally connected block spins at second order and a next-to-nearest-neighbour interaction between block spins lying along the same lattice direction at third order (see the appendix for details). It is not possible to remove both these interactions simultaneously using $\rho_0$ and $\rho_1$ alone; we must include at least one non-local operator (e.g. $R_{3,n'}[\sigma]$ which depends on spins in the blocks displaced from $n'$ by $(2, 0)$, $(-2, 0)$, etc) with a coupling strength ($\rho_3$) of the same order as the next-to-nearest-neighbour correlation. In general we find that to obtain an exactly optimised transformation we must add non-local operators with coupling strengths of the same range as the correlation functions. Furthermore, we can construct many

different exactly optimised weight functions, each describing a different behaviour close to the critical point (i.e. different critical exponents). In contrast to the one-dimensional situation no one of these weight functions is significantly more local than any of the others.

There is no evidence that a local optimised RG transformation will emerge, although the possibility cannot be ruled out that at the critical point the high-order terms conspire to cancel the leading order terms to make the transformation local. However, this calculation does demonstrate that the renormalisation group flow away from the fixed point cannot be constrained to lie in the direction of a nearest-neighbour Ising model only, as this would require a weight function of the same range as the correlation function. This conclusion has also been noted by M Luscher (in a private communication to Swendsen (see Swendsen 1986)) who pointed out that the nearest-neighbour Ising model has an isotropic correlation function *only* close to its critical point so that a transformation which confines the flow of the fixed point along the direction of the nearest-neighbour Ising model would break the symmetry of the correlation function.

After completing this work Fisher and Randeria (1986) pointed out that the fixed point cannot be made to lie anywhere on the critical surface, but is in fact unique up to displacements in the direction of redundant operators. If then the nearest-neighbour Ising model is displaced from the fixed point by irrelevant *scaling* operators then it would only be possible to make the nearest-neighbour Ising model a fixed point by using a singular (non-local) transformation—this was pointed out by Swendsen (1986) in reply to Fisher and Randeria. There is no indication from my work that anything other than this is occurring.

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**Appendix**

In this section the calculation for the two-dimensional Ising model is outlined. This calculation is done in the framework of high-temperature expansions. It is sufficient to look at the first few terms only.

For clarity we will look at a specific example of a square lattice with $M$ lattice sites and periodic boundary conditions. Furthermore we will choose to divide the lattice up into $2\times 2$ block spins as shown in figure 1. Although the detail of the following analysis depends on these choices, similar arguments to those presented will hold for different choices of lattice geometry and block spin.

Swendsen's weight function, (1.2), is not in a very convenient form for doing an analytical calculation since it is highly non-linear. A more convenient weight function to use is

$$W[\sigma', \sigma] = \prod_{n'} \tfrac{1}{2}\left(1 + \sigma'_{n'} \sum_{\alpha} \tau_{\alpha} R_{\alpha,n'}[\sigma]\right). \tag{A1}$$

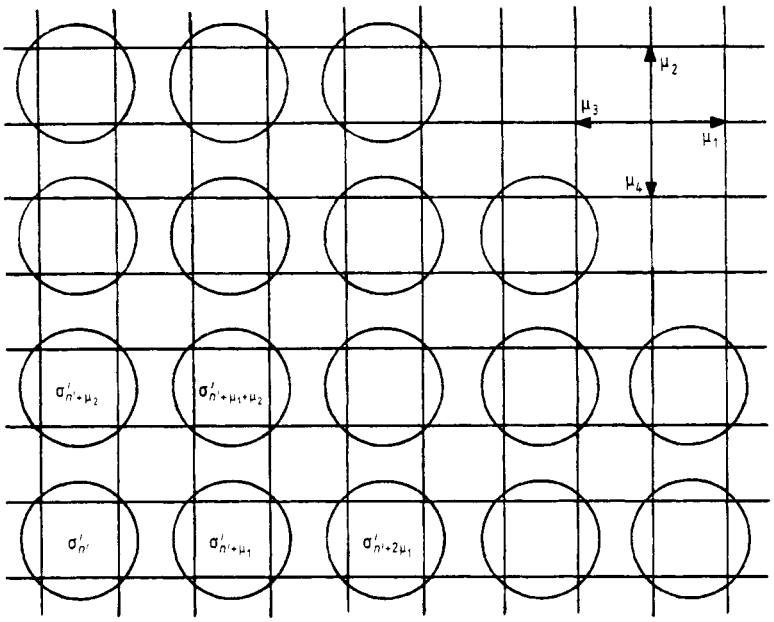


**Figure 1.** A square lattice divided into $2\times2$ block spins.

(Note that the operators are no longer linear functions of the original spins. Hence we do not have any conditions on $\tau_0$ as we did in the one-dimensional case.) Equation (A1) can be regarded as an expansion of the weight function (1.2) where (provided $\tau_1$ is small) $\tau_1=\rho_1+O(\rho_1^3)$, etc. The operators, $R_{\alpha n'}[\sigma]$, can be single-spin or multi-spin operators. The multi-spin operators are required to correct the many-spin correlation functions. Thus, for example, to make the four-spin correlations have the form of a nearest-neighbour Ising model we must introduce three-spin operators. In the following it is sufficient to restrict our attention to single-spin operators which we will denote by a Latin index (i.e. $R_{a,n'}[\sigma]$). We define the nearest-neighbour operator, $R_{1,n'}[\sigma]$, to be:

$$R_{1,n'}[\sigma]=\sum_{a=1}^{4} R_{0,n'+\mu_a}[\sigma] \tag{A2}$$

where $\mu_a$ are lattice vectors as shown in figure 1. The other single-spin operators can be defined likewise. There will be two next-to-nearest-neighbour operators: $R_{2,n'}[\sigma]$ which is the sum of the block-spin operators ($R_{0,n'}[\sigma]$) displaced from $n'$ by $(1,1)$, $(-1,1)$, $(1,-1)$ and $(-1,-1)$; and $R_{3,n'}[\sigma]$ which is the sum of the block-spin operators displaced from $n'$ by $(2,0)$ and its three symmetric partners.

To find the effective Hamiltonian we expand $\exp\{-H[\sigma]\}$ as a high-temperature series:

$$\exp(-H[\sigma])=\cosh^{2M}K\prod_{n'=1}^{M}\prod_{a=1}^{2}(1+\omega\sigma_{n'}\sigma_{n'+\mu_a}). \tag{A3}$$

It is useful to interpret this as a graphical expansion where every term represents a lattice with each link either occupied (with weight $\omega$) or unoccupied. The single-spin operators, $R_{a,n'}[\sigma]$, can be regarded as sources and the whole weight function as a sum of lattices where each block-spin site is either occupied by a source (with weight

$\tau_a$) or unoccupied. The sum over configuration has the effect of removing all graphs containing an open chain of occupied links, or an unconnected source.

Using this expansion we can now construct $\exp\{-H'[\sigma']\}$. The nearest-neighbour term $\sigma'_{n'}\sigma'_{n'+\mu}$ will have a contribution from the graph shown in figure 2($a$) of

$$\omega\tau_0^2\langle\sigma_{2n'}R_{0,n'}[\sigma]\rangle^2 \tag{A4}$$

where $\langle O[\sigma]\rangle = 2^{-M}\sum_{\text{config.},\sigma} O[\sigma]$. Form our definition of $R_{0,n'}[\sigma]$ given in (1.3) we find

$$\langle\sigma_{2n'}R_{0,n'}[\sigma]\rangle = \tfrac{3}{8} \tag{A5}$$

(Since we have defined all the single-spin operators as a sum of block-spin operators this relationship is true for all single-spin operators which overlap with a spin at some lattice site.) There are in fact two graphs which contribute to the nearest-neighbour coefficient at lowest order (the one shown in figure 2($a$) and its partner). At the next order there are six graphs which contribute (figure 2($b$) shows one of these). There is also a term contributing to the nearest-neighbour coefficient which comes from the overlap of the $R_{0,n'}[\sigma]$ operator with the $R_{1,n'}[\sigma]$ operator. Thus the lowest couple of terms contributing to the new nearest-neighbour coefficient are

$$(2\omega + 6\omega^2)\tau_0^2\langle\sigma_{2n'}R_{0,n'}[\sigma]\rangle^2 + \tau_0\tau_1\langle R_{0,n'+\mu}[\sigma]R_{1,n'}[\sigma]\rangle. \tag{A6}$$

Using the definition of $R_{1,n'}[\sigma]$ given in equation (A2) we find

$$\langle R_{0,n'+\mu}[\sigma]R_{1,n'}[\sigma]\rangle = \tfrac{5}{8}. \tag{A7}$$

Again this is true for any pair of single-spin operators which overlap with each other at some block site.

There are two next-to-nearest-neighbour terms generated by this transformation, namely $\sigma'_{n'}\sigma'_{n'+\mu_a+\mu_b}(\mu_a \perp \mu_b)$ and $\sigma'_{n'}\sigma'_{n'+2\mu}$. Graphs contributing to these terms are shown in figures 3($a$) and 3($b$), respectively. The lowest-order contributions to the

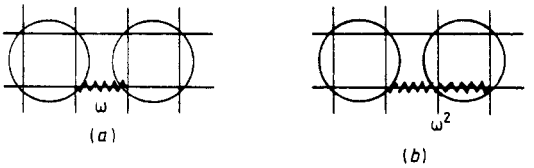


**Figure 2.** Typical graphs contributing to the nearest-neighbour term $\sigma'_{n'}\sigma'_{n'+\mu_1}$ at ($a$) first order and ($b$) second order.

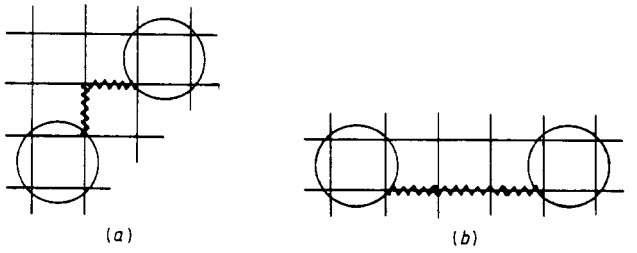


**Figure 3.** Typical graphs contributing to the two next-to-nearest-neighbour terms ($a$) $\sigma'_{n'}\sigma'_{n'+\mu_1+\mu_2}$ and ($b$) $\sigma'_{n'}\sigma'_{n'+2\mu_1}$.

$\sigma'_{n'}\sigma'_{n'+\mu_a+\mu_b}$ term are

$$2\omega^2\tau_0^2\langle\sigma_{2n'}R_{0,n'}[\sigma]\rangle^2+8\omega^2\tau_0\tau_1\langle\sigma_{2n'}R_{0,n'}[\sigma]\rangle\langle\sigma_{2n'+\mu}R_{1,n'}[\sigma]\rangle$$
$$+2\tau_1^2\langle R_{1,n'}^2[\sigma]\rangle+2\tau_0\tau_2\langle R_{0,n'+\mu_1+\mu_2}[\sigma]R_{2,n'}[\sigma]\rangle \tag{A8}$$

while for $\sigma'_{n'}\sigma'_{n'+2\mu}$:

$$2\omega^3\tau_0^2\langle\sigma_{2n'}R_{0,n'}[\sigma]\rangle^2+4\omega^2\tau_0\tau_1\langle\sigma_{2n'}R_{0,n'}[\sigma]\rangle\langle\sigma_{2n'+\mu}R_{1,n'}[\sigma]\rangle$$
$$+\tau_1^2\langle R_{1,n'}^2[\sigma]\rangle+2\tau_0\tau_3\langle R_{0,n'+2\mu}[\sigma]R_{3,n'}[\sigma]\rangle \tag{A9}$$

If we wish to make the effective Hamiltonian have the form of a nearest-neighbour Ising model then $\exp\{-H'[\sigma']\}$ must take the form:

$$\exp(-H'[\sigma'])=C\Bigg(1+\omega'\sum_{n'}\sum_{a=1}^{2}\sigma'_{n'}\sigma'_{n'+\mu_a}+2\omega'^2\sum_{n'}\sum_{a=1}^{2}\sigma'_{n'}\sigma'_{n'+\mu_a+\mu_{a+1}}$$
$$+\omega'^2\sum_{n'}\sum_{a=1}^{2}\sigma'_{n'}\sigma'_{n'+2\mu_a}+\ldots\Bigg). \tag{A10}$$

Clearly we can achieve this in a number of ways as we have two conditions and four parameters. We would like to choose the parameters so that the weight function is as local as possible. However, we cannot choose both $\tau_2$ and $\tau_3$ to be zero. One choice of parameters that will exactly optimise the transformation at leading order is

$$\begin{aligned}\tau_0&=\tfrac{4}{3}\\ \tau_1&=O(\omega^3)\\ \tau_2&=O(\omega^3)\\ \tau_3&=\tfrac{15}{8}\omega^2+O(\omega^3).\end{aligned} \tag{A11}$$

For this choice of parameters the new coupling satisfies the equation $\omega'=\frac{1}{2}\omega+\frac{3}{2}\omega^2+O(\omega^3)$. Thus at this order the fixed point coupling is given by $\omega_c=\frac{1}{3}$. The exponent $\nu$ is given by

$$\nu=\frac{\ln(d\omega'/d\omega)_{\omega=\omega_c}}{\ln 2}=0.585. \tag{A12}$$

This should be compared with the exact results $\omega_c=0.414$ and $\nu=1$.

Another set of parameters which would optimise the transformation equally well is

$$\begin{aligned}\tau_0&=4\sqrt{2}/3\omega^{1/2}\\ \tau_1&=O(\omega^{5/2})\\ \tau_2&=-3\sqrt{2}/5\omega^{5/2}-4\omega^{7/2}+O(\omega^{9/2})\\ \tau_3&=O(\omega^{9/2}).\end{aligned} \tag{A13}$$

In this case the new coupling is given by $\omega'=\omega^2+3\omega^3+O(\omega^4)$. The fixed point for this transformation is given by $\omega_c=0.434$ and critical exponent $\nu=1.36$.

If we extend this calculation to higher orders we find that we must again add terms in the weight function of the same order as the correlation function. This is not just true for $2\times2$ block spins or square lattices but will hold for many different RG transformations on regular two-dimensional lattices.

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